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TURBULENCE INTENSITY IN DILUTE TWO-PHASE FLOWS—1

EFFECT OF PARTICLE-SIZE DISTRIBUTION ON THE TURBULENCE OF THE CARRIER FLUID

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Abstract—We deal with the effect of particle size and concentration on the intensity of turbulent fluctuations in two-phase flows. An interaction between polydisperse particles and the turbulence of the carrier fluid is considered. The theoretical analysis of that process is based on Prandtl's mixing-length theory, modified to account for the peculiarities of the viscous interaction of small particles and carrier fluid as well as an effect of the admixture inertia. For a particle-laden flow with a bidisperse particle-size distribution, the intensity of turbulent fluctuations of the particles and fluid is determined as a function of the particles sizes and their mass contents. It is shown that at a fixed total mass content of the admixture, the turbulence intensity in polydisperse flow is essentially different from that in monodisperse flow. Some results on the turbulence properties of a turbulent particle-laden jet with a polydisperse admixture are obtained.

Key Words: two-phase flow, turbulence, fluctuations, particle-size distribution

1. INTRODUCTION

There have been a number of studies recently where the effect of particle size on the turbulent properties of the carrier fluid have been examined (e.g. Gore & Growe 1989; Hetsroni 1989; Rashidi *et al.* 1990a, b).

Most of these studies dealt with the effect of the particles on the structure of the turbulence in the carrier fluid, through their effect on the coherent structures in the boundary layers. It was shown that for a certain flow Reynolds number, when the particle Reynolds number was larger than a certain value, they tended to destabilize the coherent structures and cause an increase in the frequency of ejections within a burst. Particles with smaller Reynolds number caused a decrease in the ejection frequency. The bursting frequency remained roughly constant.

Other experimental evidence (e.g. Tsuji & Morikawa 1982; Tsuji *et al.* 1984; Michaelides & Stock 1989; Michaelides *et al.* 1991; Mizukami *et al.* 1992) focused on the modulation of the turbulence of the carrier fluid in vertical tubes. Here the evidence is that particles with a Reynolds number larger than some critical value cause an increase in the intensity of turbulence, maybe mostly due to vortex shedding. Smaller particles tend to damp the turbulence, maybe by increasing the apparent viscosity.

Though the experimental evidence becomes more prevalent, there is still a paucity of data to allow for a clear mechanistic modeling of the phenomena. It is clear that the particles–turbulence interaction is a highly complex phenomenon, and it depends on the flow Reynolds number, the contents of the admixture, the physical properties, the nature of the flow, the length scales of the turbulence and the particles etc. All previous theoretical studies were limited to monodisperse particles, i.e. particles of a single/uniform size, a case which exists neither in nature nor in engineering.

Here we examine the effect of polydisperse particles on the turbulence of the carrier fluid. Naturally, this is of practical importance since we can, most likely, affect various processes by changing the size of the particles, or using various loadings of different sizes. We study the effect of the various size distributions on the turbulence of the main stream by generalizing the approach of Abramovich (1970), who did a similar analysis for monodisperse particles. During the last 20 years this approach has been used successfully to solve a number of problems of the theory of turbulent two-phase flows. First, "simple" types of two-phase flows were studied neglecting the "external" forces. Further, more complicated problems have been tackled. In particular, the gravity effect of the fluctuations structure was studied (Girshovich & Leonov 1979); the effect of the average velocity difference of particles and carrier fluid (Frishman 1979); the effect of particle diffusion (Abramovich & Girshovich 1973) etc. The results of these invesigations have been generalized in the monographs by Abramovich *et al.* (1984) and Shraiber *et al.* (1990).

This theory is based on a mixture-length theory, similar to the one suggested by Prandtl (1925). Actually, although the theory is quite old it shows a remarkable insight into the physics of turbulent flows, since Prandtl's mixing length is the distance in the transverse direction which must be traveled by a lump of fluid with its original mean velocity in order to make the difference between its velocity and the velocity in the new location equal to the mean transverse fluctuation in turbulent flow. It seems that Prandtl was modeling coherent structures long before they were discovered!

2. THE ANALYSIS

Consider a steady, incompressible turbulent flow laden with polydisperse particles. The particles' diameters are $d_1, d_2, \ldots, d_n$, which are relatively small, such that the effect of turbulent wakes can be neglected, i.e. particle Reynolds number $\mathrm{Re}_p < 110$). The particles are assumed to be in a fluid element as a structure, i.e. they persist together during its lifetime.† However, the particle and fluid element move with different velocities depending on their mass, fluid viscosity etc. The number of particles, of each diameter, in that fluid element are $k_1, k_2, \ldots, k_n$, respectively.

Further we will consider flows where transverse pressure gradients are absent (boundary layers, pipe flows). We also restrict that analysis to flows with slightly changing longitudinal velocity where the relative average velocity of the carrier fluid and the particles is negligibly small (developed turbulent flow in a jet etc.).

The mass of the particles in the ith fraction is $M_{pi} = m_i k_i$, where m_i is the mass of a single particle. The mass of all the particles in the fluid element is $M_p = \Sigma_{i=1}^{n} m_i k_i$, and it is assumed to be constant over the carrier fluid lifetime, i.e. particles do not leave the fluid element as long as this fluid element persists as an entity. This assumption is fairly good for a number of real two-phase flows (appendix C).

In accordance with Prandtl's fundamental idea of mixing length we assume, that the fluid element acquires at time $t = 0$, momentum $J_0 = m_1 v_0'$ under the influence of a hydrodynamic field disturbance (pressure fluctuation). Further, it moves by inertia and interacts with the particles inside. Then the carrier fluid and particles are described by the following equations of conservation of momentum:

$$d\left[v' + \sum_{i=1}^{n} \gamma_i v_p'^{(i)}\right] = 0 \qquad [1]$$

and

$$m_i \frac{\mathrm{d}v_p'^{(i)}}{\mathrm{d}t} = \tfrac{1}{2} C_D^{(i)} f_i \rho \tilde{v}_i |\tilde{v}_i|, \qquad [2]$$

where v' is the fluctuational velocity of the carrier fluid and $v_p'^{(i)}$ are the fluctuational velocities of the particles in fraction i (namely those which have a diameter d_i); M is the mass of the fluid in the fluid element $\gamma = M_p/M$ is the mass content of the particles in the fluid element and $\gamma_i = M_{pi}/M$ are the mass contents of the particles of fraction i, such that $\gamma = \Sigma_{i=1}^{n} \gamma_i$; C_D is the drag coefficient; f_i are the cross sections of the particles; $\tilde{v}_p^{(i)} = v' - v_p'^{(i)}$ are the relative velocities between the carrier fluid and the particles of fraction i; and ρ is the density of the fluid.

†We use the terminology adopted in Schlichting (1979): fluid particle or fluid element.

Note that on the LHS of [1] we omitted the term accounting for a drag force imposed by the surrounding fluid on the given fluid element. The latter means that we assume that the fluid element moves in inviscid fluid. It is known that this approximate is fairly good for relatively large eddies where the drag/inertia forces ratio is small.† We can use this approximation because of the fact that the size of the fluid element is much larger than that of a particle. Note also, that on the LHS of [2] we omitted the terms accounting for the nonsteady character of the particle motion in the fluid element (inertial force due to virtual mass, Basset force etc.). These effects are negligibly small in particle-laden gas flows (Boothroyd 1971; Nigmatulin 1990). For example, in a submerged particle-laden jet with the parameters $\gamma = 0.1$, $d_p = 5 \cdot 10^{-5}$ m, $\bar{\rho} = 8 \cdot 10^2$ ($\bar{\rho} = \rho_p/\rho$ is the phase density ratio), $\nu = 10^{-5}\,\mathrm{m^2/s}$ and $v'_0 = 1$ m/s, the pressure/drag force ratio, the force due virtual mass/drag force ratio, the Basset force/drag force ratio and the gravity/drag forces ratios are equal to 10^{-2}, 10^{-2}, 10^{-1} and 10^{-1}, respectively.

Equation [2] is merely a statement that the change of momentum of the particles is a result of the drag force of the fluid.

It is emphasized that the approach used to describe the particles' and the fluid fluctuational velocities does not omit pressure effects. They are accounted for by the initial momentum of the fluid element, $J_0 = m_1 v'_0$, which determines the characteristic scales of the fluid and particle velocities.

The initial conditions for [1] and [2] are

$$@\ t = 0, \qquad v' = v'_0, \quad v'^{(i)}_p = 0. \qquad [3]$$

They correspond to the physical mechanism of the fluctuational motion in turbulent two-phase flows. First, the carrier fluid fluctuation begins, then the momentum is transferred to the particles due to the drag. Thus, the reason for the fluctuational motion of the particles is the fluctuational motion of the carrier fluid. Therefore, under the conditions of finite speed of propagation of perturbations and finite relaxation times (which is the case), the fluctuational particle velocity at $t = 0$ (the moment of eddy "birth") should be equal to zero. It is emphasized that this assumption leads to physically reasonable results as well as to fairly good agreement with experimental data (Abramovich *et al.* 1984).

Integrating [1] we obtain, in dimensionless form,

$$\bar{v}' + \sum_{i=1}^{n} \gamma_i \bar{v}'^{(i)}_p = 1, \qquad [4]$$

where

$$\bar{v} = \frac{v'}{v'_0} \quad \text{and} \quad \bar{v}'^{(i)}_p = \frac{v'^{(i)}_p}{v'_0}.$$

(the fluctuational velocities are normalized by the initial fluid fluctuation). Equation [4] can also be expressed as

$$\bar{v}' + \sum_{i=1}^{n} [\gamma_i(\bar{v}'^{(i)}_p - v') + \gamma_i \bar{v}'] = 1$$

or

$$\bar{v}' = (1+\gamma)^{-1}\left[1 + \sum_{i=1}^{n} \gamma_i \tilde{v}^{(i)}_p\right]. \qquad [5]$$

Equation [5] states that the fluctuational velocity of the carrier fluid (normalized to its initial value) is determined by the total mass content of the particles γ, and by the relative velocities of the particles of various fractions $\tilde{v}^{(i)}_p$. The former determines the inertial characteristics of the admixture, while the latter expresses the viscous interaction (drag) between the particles and the carrier fluid.

†The estimates show that in real turbulent jet flows this ratio is small enough—of the order of 0.03–0.15 for submerged particle-laden jet with the following parameters: $\gamma = 0.01$ to 0.1, $d_p = 5 \cdot 10^{-5}$ m, $\bar{\rho} = 8 \cdot 10^2$, $\nu = 10^{-5}\,\mathrm{m^2/s}$.

One can examine some extreme cases—say, the case when the particles are very fine ($d \to 0$) and the relative velocities are negligible, $\tilde{v}_p^{(i)} = 0$, which leads to the minimal fluctuational velocities of the carrier fluid:

$$\bar{v}' = (1+\gamma)^{-1}.$$

Assume some form of a drag force, say the Stokes law

$$C_D^{(i)} = \frac{24}{\mathrm{Re}_i}, \quad [6]$$

where the Reynolds number is defined based on the relative velocity:

$$\mathrm{Re}_i = \frac{\tilde{v}_p^{(i)} d_i}{\nu}. \quad [7]$$

Here $\nu = \mu\rho^{-1}$; μ is the viscosity of the fluid. Then [2] may be rewritten as

$$\frac{\mathrm{d}v_p'^{(i)}}{\mathrm{d}t} = \tfrac{1}{2}\rho \frac{24}{\tilde{v}_p^{(i)} d_i} \nu \tilde{v}_p^{(i)} |\tilde{v}_p^{(i)}| \frac{f_i}{m_i}$$

or simply

$$\frac{\mathrm{d}v_p'^{(i)}}{\mathrm{d}t} = \frac{18\mu\tilde{v}_p^{(i)}}{d_i^2 \rho_i}. \quad [8]$$

Combining [4] and [8] we get

$$\frac{\mathrm{d}\bar{v}_p'^{(i)}}{\mathrm{d}t} = \tau_i^{-1}\left[1 - \sum_{i=1}^{n} \gamma_i \bar{v}_p'^{(i)} - \bar{v}_p'^{(i)}\right], \quad [9]$$

where the relaxation time is defined as

$$\tau_i = \frac{d_i^2 \rho_i}{18\mu}. \quad [10]$$

Equations [4] and [9], with the initial condition [3] describe the turbulent fluctuations of the particles of various fractions i, and of the carrier fluid.

3. BIDISPERSE ADMIXTURE

Consider a bidisperse admixture, i.e. a carrier fluid with particles of two sizes in it. Then [4] and [9] read as follows:

$$\bar{v}' + \gamma_1 \bar{v}_1' + \gamma_2 \bar{v}_2' = 1, \quad [11]$$

$$\frac{\mathrm{d}\bar{v}_1'}{\mathrm{d}t} = \tau_1^{-1}[1 - (1+\gamma_1)\bar{v}_1' - \gamma_2 \bar{v}_2'] \quad [12]$$

and

$$\frac{\mathrm{d}\bar{v}_2'}{\mathrm{d}t} = \tau_2^{-1}[1 - \gamma_1 \bar{v}_1' - (1+\gamma_2)\bar{v}_2'], \quad [13]$$

where $\bar{v}_1' = \bar{v}_p'^{(i)}$, $\bar{v}_2' = \bar{v}_p'^{(2)}$ and indexes 1 and 2 refer to the large and small particles, respectively. The initial conditions [3] are reduced to:

$$@\ t = 0 \qquad \bar{v}' = 1, \quad \bar{v}_1' = \bar{v}_2' = 0. \quad [14]$$

One can examine some general properties of the solutions to the system [11]–[13] with [14]. Eliminating the time from [12] and [13] we get

$$\frac{\mathrm{d}\bar{v}_1'}{\mathrm{d}\bar{v}_2'} = \omega \frac{1 - (1+\gamma_1)\bar{v}_1' - \gamma_2\bar{v}_2'}{1 - \gamma_1\bar{v}_1' - (1+\gamma_2)\bar{v}_2'}, \quad [15]$$

where $\omega = d_2^2/d_1^2$ ($0 < \omega \leqslant 1$; $\omega = 1$ corresponds to flow with monodisperse particles). Integrating [15], one gets the following relations between $\bar{v}_1'$ and v_2' (details are given in appendix A):

$$x[a + (b - \alpha)z - \beta z^2]^{1/2}\left[\frac{-2\beta z + (b - \alpha) - \sqrt{\xi}}{-2\beta z + (b - \alpha) + \sqrt{\xi}}\right]^{-(b+\alpha)/2\sqrt{\xi}} = c, \qquad [16]$$

where $x = \bar{v}_2' - (1 + \gamma)^{-1}$, $z = y/x$ ($1 \leqslant z < \infty$; $z = 1$ corresponds to the flow of a monodisperse admixture), $y = \bar{v}_1' - (1 + \gamma)^{-1}$, $a = -\gamma_2\omega$, $b = -(1 + \gamma_1)\omega$, $\beta = -\gamma_1$, $\alpha = -(1 + \gamma_2)$, $\xi = (b - \alpha)^2 + 4\beta a$ and

$$c = -\sqrt{\frac{1-\omega}{1+\gamma}}\left[\frac{2\gamma_1 - (1 + \gamma_1)\omega + (1 + \gamma_2) - \sqrt{\xi}}{2\gamma_1 - (1 + \gamma_1)\omega + (1 + \gamma_2) + \sqrt{\xi}}\right]^{-(b+\alpha)/2\sqrt{\xi}}.$$

There are interesting cases which can be considered, resulting in significant simplifications of [16]. Naturally, [16] can be solved numerically for any case—but here we want just to look at an example, to demonstrate the plausibility of the solution. For example, consider an admixture with bidisperse particles, such that $d_1 \gg d_2$ (i.e. $\omega \ll 1, z \gg 1$), then [16] reduces to

$$\bar{v}_1' = (1 + \gamma)^{-1} + c\gamma_1^{-1/2} \qquad [17a]$$

and

$$\bar{v}_2' = (1 + \gamma)^{-1}. \qquad [17b]$$

These expressions, together with [11], enable one to calculate the fluctuational velocity of the carrier fluid in this particular case,

$$\bar{v}' = 1 - \gamma(1 + \gamma)^{-1} - c\gamma_1^{1/2}; \qquad [18]$$

where c, defined above, can also be simplified if $\gamma_1 < \gamma_2$:

$$c \approx -\gamma_1^{1/2}(1 + \gamma)^{-1}.$$

Therefore,

$$\bar{v}_1' \approx 0, \quad \bar{v}_2' = (1 + \gamma)^{-1}, \quad \bar{v}' = (1 + \gamma)^{-1}(1 + \gamma_1). \qquad [19]$$

This result implies that the fluctuation intensity of the small particles (2) is independent of the mass content of the large particles (1). Also, for a fixed total mass content of the admixture γ, the fluctuation intensity of the carrier fluid is proportional to the mass content of the large particles γ_1, which was shown experimentally to be qualitatively correct.

Equation [16] gives the relation between the intensity of the velocity fluctuations of particles of various sizes, but it does not actually give the absolute values of these fluctuations. In order to obtain them one needs to integrate [12] and [13] (appendix B), with the result:

for the large particles,

$$\bar{v}_1' = (1 + \gamma)^{-1}\{-(\varphi_2 - \varphi_1)^{-1}[\gamma_1^{-1}(1 + \gamma) - \varphi_2][\gamma_1^{-1}(1 + \gamma) - \varphi_1]\exp(-\gamma_1\varphi_1\bar{\tau}^*) + (\varphi_2 - \varphi_1)^{-1}[\gamma_1^{-1}(1 + \gamma) - \varphi_1][\gamma_1^{-}(1 + \gamma) - \varphi_2]\exp(-\gamma_1\varphi_2\bar{\tau}^*) + 1\}; \qquad [20]$$

and

for the small particles,

$$\bar{v}_2' = (1 + \gamma)^{-1}\{(\varphi_2 - \varphi_1)^{-1}[\gamma_1^{-1}(1 + \gamma) - \varphi_2]\exp(-\gamma_1\varphi_1\bar{\tau}^*) - (\varphi_2 - \varphi_1)[\gamma_1^{-1}(1 + \gamma) - \varphi_1]\exp(-\gamma_1\varphi_2\bar{\tau}^*) + 1\}; \qquad [21]$$

where

$$\varphi_{1,2} = 0.5\{\gamma_1^{-1}[(1 + \gamma_1)\omega + (1 + \gamma_2)] \pm \sqrt{\gamma_1^{-2}\{[(1 + \gamma_2) - (1 + \gamma_1)\omega]^2\} + 4\omega\gamma_1^{-1}\gamma_2}\}$$

and $\bar{\tau}^* = t^*/\tau_2$, t^* is the time of interaction of the particle and carrier fluid. The velocity of the carrier fluid $\bar{v}'$ must be computed from [11].

In two-phase flows the fluid element, loaded by a polydisperse admixture, persists as an entity until the largest particle leaves it (the largest particle leaves it first). The distance which

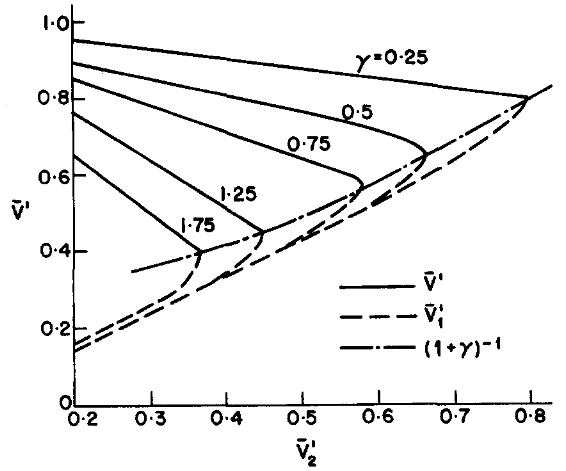

Figure 1. Turbulence intensity of the carrier fluid and the large particles as a function of the turbulence intensity of the small particles, for different mass contents of the polydisperse admixture ($\omega = 0.8, \gamma_1 = \gamma_2 = \gamma/2$).

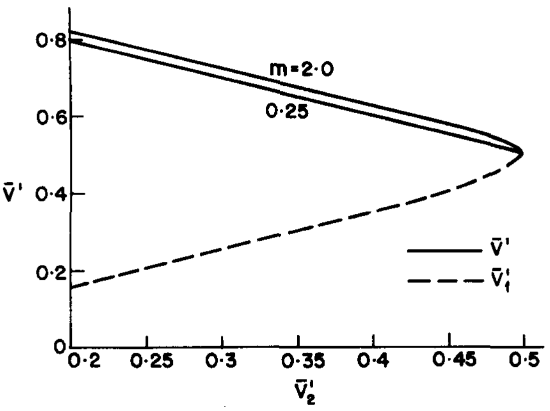

Figure 2. Turbulence intensity of the carrier fluid and the large particles as a function of the turbulence intensity of the small particles, for different ratios of the mass contents of the large and small particles ($\gamma = 1, \omega = 0.8$).

this particle travels during the interaction time may be estimated as ($l \geqslant l_1 \geqslant l - d_f/2$). Here l and l_1 are the fluid element and particle mixing lengths, respectively; d_f is a characteristic size of the fluid element. Therefore, we can write the following expression for the interaction time: $t^* = l_1/\bar{v}_1' = l/\bar{v}_1'(1 - d_f/2l)$, where $\bar{v}_1'$ is an average velocity during the interaction time. Under the condition $d_f/2l \ll 1$ we obtain $t^* = l_1/\bar{v}_1'$. The dimensionless time $\bar{\tau}^*$, i.e. the ratio of the characteristic time t^* to the relaxation time τ_2 is equal to

$$\bar{\tau}^* = 36\frac{\bar{l}}{\bar{v}_1' \cdot \bar{\rho} \cdot \mathrm{Re}_p} = \frac{\tau^*}{\bar{v}_1'}, \qquad [22]$$

where $\bar{\rho} = \rho_p/\rho$, ρ_p and ρ are density of the particles and the fluid, respectively, $\bar{l} = l/d_2$ and $\tau^* = 36(\bar{l}/\bar{\rho} \,.\, \mathrm{Re}_p)$, $\mathrm{Re}_p = (d_2 \cdot |v_0'|)/\nu$.

If the interaction time is significantly larger than the relaxation time, $\tau^* \gg 1$, i.e. when particles are following the fluid's fluctuations closely, the fluctuational velocities of all the particles tend to the limit $(1 + \gamma)^{-1}$. For finite values of τ^*, the level of the fluctuations of the particles and the fluid depends on the ratio of particle diameters, the total mass content of the admixture, the mass content of the fine and coarse particles, the ratio of particles densities, the particle Reynolds number, the physical properties of the carrier fluid and the particles as well as the ratio of the characteristic scale of turbulence and the fine particle size.†

4. RESULTS AND DISCUSSION

In figures 1–3 the dependences of the dimensionless fluctuation intensities of the carrier fluid, $\bar{v}'$, and the particles, $\bar{v}_1'$ and $\bar{v}_2'$, are depicted as computed by [11] and [16]. Figure 1 depicts the fluctuation intensity of the carrier fluid, $\bar{v}'$ (solid lines), and the large particles, $\bar{v}_1'$, as a function of the fluctuation intensity of the small particles, $\bar{v}_2'$, and the total mass content as a parameter. It can be observed that an increase in the fluctuation intensity of the smaller particles, $\bar{v}_2'$, is accompanied by a decrease in the fluctuation intensity of the carrier fluid and an increase in the fluctuational velocity of the large particles, $\bar{v}_1'$. Also, at some value of $\bar{v}_2'$ the curves of $\bar{v}'$ and $\bar{v}_1'$ intersect (at some value of γ). At this point the fluctuation intensities of particle of both sizes and of the carrier fluid are equal. At this point the fluctuation intensities depend only on the total loading, as $\bar{v}_2' = \bar{v}_1' = \bar{v}' = (1 + \gamma)^{-1}$. This also corresponds to the case when $\tau^* \gg 1$, and all the particles follow the fluid very closely. Note, that an increase in the fluctuational velocity of the

†Note that the mixing length in two-phase flow depends on the admixture inertia. However, the effect of this dependence on velocity fluctuations is small under the condition $\tau^* > 1$. Therefore, we do not account for this effect when estimating the fluctuations intensity.

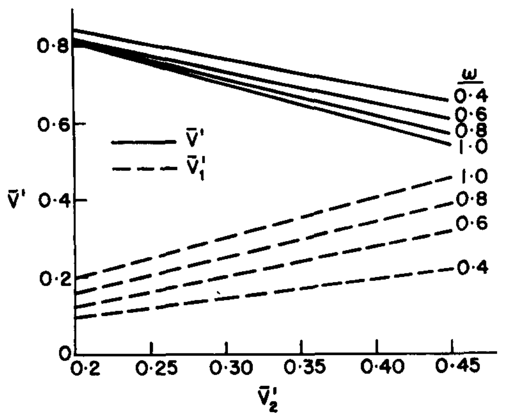

Figure 3. Turbulence intensity of the carrier fluid and the large particles as a function of the turbulence intensity of the small particles for different ratios of the diameters of the large and small particles ($\gamma_1 = \gamma_2 = \gamma/2 = 0.5$).

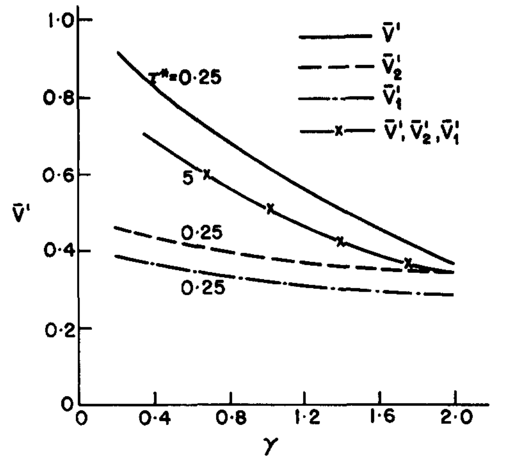

Figure 4. Dependence of the turbulent quantities of the polydisperse mixture on the total mass content of the admixture ($\gamma_1 = \gamma_2 = \gamma/2$, $\omega = 0.8$, $\tau^* = 0.25$).

particles due to a decrease in the fluctuational velocity of the carrier fluid is related to the momentum transfer from fluid to particle as a result of viscous interaction (the total momentum of the system fluid element with particles is conserved during a period equal to one turbulent integral time scale).

The dependence of the velocity fluctuation intensity of the carrier fluid, $\bar{v}'$, and the large particles, $\bar{v}'_1$, on $\bar{v}'_2$ for various ratios of the mass content of large particles to small particles, $m = m_1/m_2$ (with a fixed total mass content), is depicted in figure 2. The velocity fluctuations of the carrier fluid increase only slightly when increasing m from 0.25 to 2.0. This is due to a decrease in the mass content of small particles, and the corresponding decrease in energy dissipation due to their presence.

The dependence of $\bar{v}'$ and $\bar{v}'_1$ on $\bar{v}'_2$ for various ratios of particles' diameter ($\omega = d_2^2/d_1^2$) are depicted in figure 3. The variation is almost linear. Again it is seen that the fluctuation velocity of the carrier fluid decreases with an increase in the fluctuation velocity of the small particles, while that of the larger particles increases. These effects are more pronounced as ω gets closer to unity. The dependences of the intensity of the velocity fluctuations of the carrier fluid, $\bar{v}'$, of the small particles, $\bar{v}'_2$, and of the larger particles, $\bar{v}'_1$, on the total mass content of the admixture are depicted in figure 4. An increase in the mass content, γ, is always followed by a decrease in fluctuations. When the value of the parameter τ^* increases, the curves for $\bar{v}'(\gamma)$, $\bar{v}'_1(\gamma)$ and $\bar{v}'_2(\gamma)$ converge, and at $\tau^* = 5$ they practically merge.

In figure 5 one observes that a change in the ratio of the mass content of large particles to small ones, $m = m_1/m_2$, for a constant value of total mass content, does not affect the level of the fluctuations. An increase in the loading of large particles leads to a small increase in the turbulence intensity of the carrier fluid, for $0.25 < \tau^* < 5$.

The ratio of the small-to-large particle size, $\omega = d_2^2/d_1^2$, affects the turbulence intensity of the carrier fluid and particles, as is evidenced from figure 6.

It is shown that the effect of the parameter ω on the turbulent fluctuations of the particles of various sizes is significant. When τ^* is fixed, a decrease in the parameter ω leads to an increase in the turbulent fluctuations of the fine particles and an increase in those of the coarse cnes.

The dependence of the carrier fluid turbulence on the diameter ratio is depicted in figure 7. The results for the turbulence intensity in the polydisperse and monodisperse systems, with equal total mass content of the admixture, are shown in figure 7 as the ratio of

$$\frac{v'_{\mathrm{pol}} - v'_{\mathrm{mon}}}{v_{\mathrm{mon}}} \cdot 100\% \text{ vs } \omega$$

(v'_{pol} and v'_{mon} are the carrier fluid fluctuations in the polydisperse and monodisperse systems, respectively). Curve 1 corresponds to the case when d_2 and τ^* are constant and d_1 is varied; curve

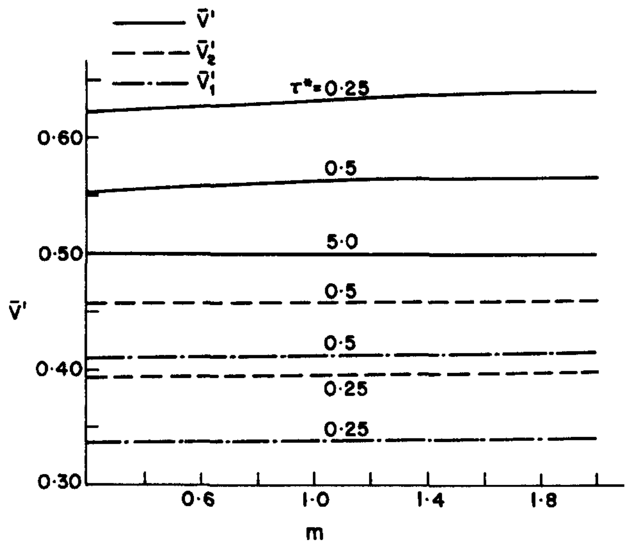

Figure 5. Dependence of the turbulent quantities of the polydisperse mixture on the ratio of the mass contents of the large and small particles ($\omega = 0.8, \gamma = 1.0$).

2 corresponds to the case when d_2 and τ^* vary and d_1 is constant. The bounding point $\omega = 1$, at the right end of curves 1 and 2, corresponds to a monodisperse system. It is seen that the turbulence intensity in the polydisperse system differs significantly from that in the monodisperse one.

It is emphasized that the turbulence in the polydisperse system may be either higher or lower than that in the monodisperse one. For example, if the size of the coarse particles in the polydisperse system is equal to those in the monodisperse one (whereas the size of the fine particles is smaller than those in the monodisperse admixture) the turbulence intensity in the polydisperse system is lower than that in the monodisperse one. In this case a decrease in the fine particles size leads (at fixed, γ, γ_1 and γ_2) to a larger deviation of v'_{pol} from v'_{mon}. The latter effect is related to an increase in the energy spent on the acceleration of the admixture, since a decrease in the fine particles size leads to an increase in their velocity fluctuations. An increase in the coarse particles size is accompanied by a decrease in the spent energy, which leads to the increase in the turbulence intensity of the carrier fluid.

The effects predicted above are of great importance when we try to understand the peculiarities of turbulent particle-laden flows. Naturally, these effects should be verified by experiments. The present theory might be used in new experimental investigations of the fluctuational characteristics of polydisperse flows.

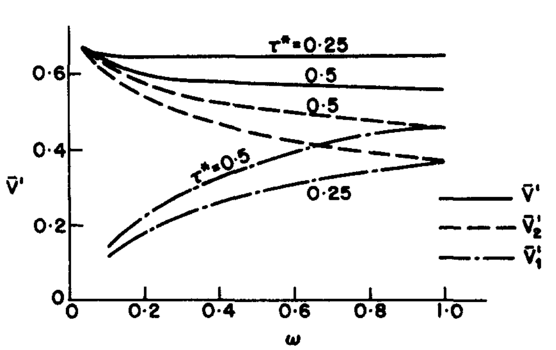

Figure 6. Dependence of the turbulent quantities of the polydisperse mixture on the ratio of the diameters of the large and small particles ($\gamma_1 = \gamma_2 = \gamma/2 = 0.5$).

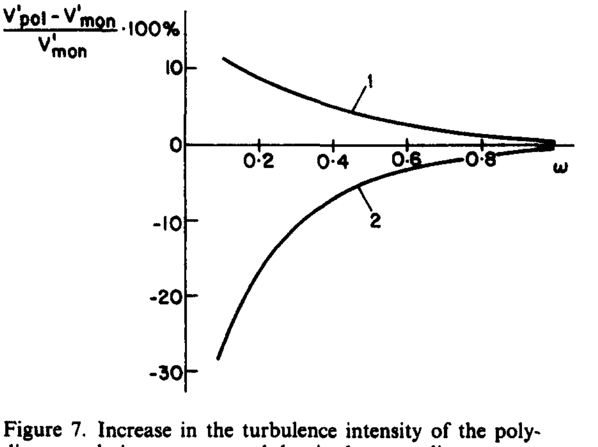

Figure 7. Increase in the turbulence intensity of the polydisperse admixture compared that in the monodisperse one, vs the particles' diameter ratio: (1) $d_2 = \text{const}$, $\tau^* = \text{const}$, $d_1 = \text{var}(\gamma_1 = \gamma_2 = \gamma/2 = 0.5, \epsilon = 0.5)$; (2) $d_1 = \text{const}$, $d_2 = \text{var}$, $\tau^* = \text{var}(\gamma_1 = \gamma_2 = \gamma/2 = 0.5)$.

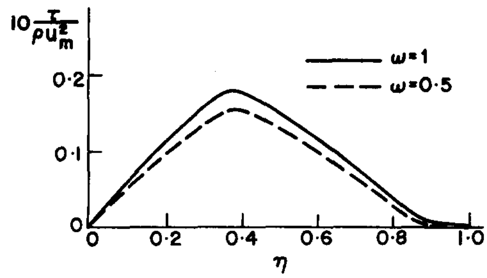

Figure 8. Radial variation of the turbulent shear stress in monodisperse and polydisperse two-phase flow in a submerged jet: (——) jet with the monodisperse admixture; (---) jet with the polydisperse admixture.

5. FLUCTUATIONS IN A SUBMERGED JET

Now we consider an effect of a polydisperse admixture on the turbulent properties of a submerged jet, by using [11] and [20]–[22]. Let us rearrange the RHSs of [20] and [21] to the following form:

$$\frac{v_1'}{u_m}=\frac{v_0'}{u_m}F_1;\quad \frac{v_2'}{u_m}=\frac{v_0'}{u_m}F_2;\quad \frac{v'}{u_m}=1-\frac{v_0'}{u_m}(\gamma_1F_1+\gamma_2F_2); \qquad [23]$$

where $F_{1,2}=F_{1,2}(\gamma,\tau^*,v_0')$ and u_m is the mean velocity at the jet axis. The multiplier v_0'/u_m in [23] is a function of the exit velocity, the diameter of the jet and the location. According to Prandtl's hypothesis, $v_0'=l(\mathrm{d}u/\mathrm{d}y)$. Considering as usual $u/u_m=f(\eta)$, we obtain

$$\frac{v_0'}{u_m}=\frac{l}{\delta}f'(\eta). \qquad [24]$$

where $\eta=y/\delta$ is the nondimensional lateral coordinate measured from the jet centerline, δ is the jet width and $f(\eta)$ is a function of η, which may be described (Abramovich *et al.* 1984) as:

$$u_m=\left(\frac{K_0}{2\pi\cdot 0.067}\right)^{1/2}\frac{1}{\delta},\quad \delta=27.3\beta^2x;\quad f(\eta)=(1-\eta^{3/2})^{1/2}. \qquad [25]$$

Here $K_0=I_0/\rho$, I_0 is the total jet momentum in the axial direction x and $\beta=0.09$. The distribution of the turbulent viscosity and turbulent shear stress in the flow field of a submerged jet may be determined by using the known correlations of turbulence theory,

$$\nu_T=l\overline{|v'|}\quad\text{and}\quad \tau_T\approx\rho\overline{u'v'}(\overline{u'v'}=\overline{|u'|}\,\overline{|v'|}), \qquad [26]$$

where τ_T is the turbulent shear stress; ν_T is the turbulent viscosity. Equations [25] and [20]–[22] can be used to compute the fluctuational velocity in a two-phase jet. In figure 8 such data are depicted for two-phase jets with exit diameter $d=2\times10^{-2}$ m, velocity $u_0=70$ m/s and air viscosity $\nu=10^{-5}\,\mathrm{m^2/s}$. The particles in the jet have diameter $d_2=5\cdot10^{-5}$ m and density ratio $\rho_p/\rho_f=2\cdot10^3$. In the figure, the turbulent shear stress is plotted vs the dimensionless distance from the jet axis, at a location downstream from the exit (nozzle) $x/d=22.5$, for a polydisperse admixture ($\omega=0.5$) and a monodisperse one ($\omega=1.0$). It is clear that the turbulent shear stress in the two-phase jet with a polydisperse admixture is smaller than in that with a monodisperse one.

Note that the calculation of the turbulent quantities in the two-phase submerged free jet by using [20]–[22] does not specify a model of turbulence. To determine v_0' in the two-phase jet we may use any model of turbulence or experimental data on turbulent fluctuations in the pure gas jet.

6. CONCLUSIONS

By using the mixing-length theory we proposed a simple model for the calculation of turbulence intensity in two-phase flows with a polydisperse admixture. For flows with a bimodal admixture (particles of two sizes) the characteristics of the turbulent fluctuation fields have been calculated. As an example, a two-phase submerged jet with a bidisperse admixture has been considered. It was shown for a fixed total mass content of the admixture the turbulence intensity in a polydisperse flow is essentially distinct from that in a monodisperse one.

The following results have been obtained:

1. The turbulence intensity in the bimodal polydisperse system (carrier fluid and particles of two fractions) is determined by the following parameters: the total mass content of the admixture, the mass contents of the particles of the fine and coarse fractions, their diameter ratio, the particles and carrier fluid density ratio, the particle Reynolds numbers and the ratio of the mixing length to the diameter of the particles in one group.
2. An increase in the total mass content of a polydisperse admixture reduces the turbulence intensities of the carrier fluid and the particles.
3. The turbulence intensity in the polydisperse system may be higher or lower than that in the monodisperse one, depending on the ratio of the particles sizes and their mass contents.
4. In the case of equal total mass contents in the polydisperse and monodisperse systems, the turbulence intensity of the former is higher when the coarse particles' diameter in the polydisperse system (bimodal admixture with equal mass contents of fine and coarse particles) is larger than the particles' diameter in the monodisperse system.
5. The turbulence intensity of the polydisperse system is lower than that of the monodisperse one when the diameter of its coarse particles is equal to the diameter of particles in the monodisperse system, given that the total mass contents in the systems are equal and that in the polydisperse system the coarse and fine particles have equal total mass content.
6. When the other parameters are fixed, a decrease in the fine particles' diameter or increase in their content in the admixture leads to a reduction in the turbulence intensity in the polydisperse system. An increase in the coarse particles' content or diameter is accompanied by a turbulence intensity increase in the polydisperse system, with the other parameters being fixed.

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APPENDIX A

Write [15] as follows:

$$\frac{d\bar{v}_1'}{d\bar{v}_2'} = \frac{a\bar{v}_2' + b\bar{v}_1' + \Omega_*}{\alpha\bar{v}_2' + \beta\bar{v}_1' + \Omega}, \qquad [A1]$$

where

$$a = -\gamma_2\omega; \quad b = -(1+\gamma)\omega; \quad \Omega_* = \omega; \quad \alpha = -(1+\gamma_2); \quad \beta = -\gamma_1; \quad \Omega = 1.$$

Let $\bar{v}_2' = x + \epsilon_2$ $\bar{v}_1' = y + \epsilon_1$. Then the numerator and denominator on the RHS of [A1] may be arranged in the form

$$a\bar{v}_2' + b\bar{v}_1' + \Omega_* = ax + by + (a\epsilon_2 + b\epsilon_1 + \Omega_*) \qquad [A2]$$

and

$$\alpha\bar{v}_2' + \beta\bar{v}_1' + \Omega = \alpha x + \beta y + (\alpha\epsilon_2 + \beta\epsilon_1 + \Omega). \qquad [A3]$$

Since the values of ϵ_1 and ϵ_2 are arbitrary, one can choose them to satisfy the equalities $(a\epsilon_2 + b\epsilon_1 + \Omega^*) = 0$ and $(\alpha\epsilon_2 + \beta\epsilon_1 + \Omega) = 0$. Therefore:

$$\epsilon_1 = \frac{a - \alpha\omega}{\alpha b - a\beta} = \frac{1}{1+\gamma}; \quad \epsilon_2 = \frac{b - \beta\omega}{a\beta - \alpha b} = \frac{1}{1+\gamma}.$$

Taking into account [A2] and [A3], we write [A1] in the following form:

$$\frac{dy}{dx} = \frac{ax + by}{\alpha x + \beta y}. \qquad [A4]$$

Defining a new variable $z = y/x$ $(y' = z'x + z)$, we obtain

$$z'x + z = \frac{a + bz}{\alpha + \beta z}. \qquad [A5]$$

Separating variables in [A5], we obtain

$$\frac{\alpha + \beta z}{a + (b - \alpha)z - \beta z^2}\,dz = \frac{dx}{x}. \qquad [A6]$$

Since the value of Δ is negative,

$$\Delta = -4\beta a - (b - \alpha)^2 = -4\gamma_1\gamma_2\omega - [-(1-\gamma_1)\omega + (1+\gamma_2)]^2 < 0,$$

and integration of [A6] yields

$$\frac{\alpha}{\sqrt{-\Delta}}\ln\frac{-2\beta z+(b-\alpha)-\sqrt{-\Delta}}{-2\beta z+(b-2)+\sqrt{-\Delta}}+\beta\left\{-\frac{1}{2\beta}\ln[a+(b-\alpha)z-\beta z^2]\right.$$
$$\left.+\frac{b-\alpha}{2\beta}\frac{1}{\sqrt{-\Delta}}\ln\frac{-2\beta z+(b-\alpha)-\sqrt{-\Delta}}{-2\beta+(b-\alpha)+\sqrt{-\Delta}}\right\}=\ln x+\ln c. \quad \text{[A7]}$$

Rearranging [A7] we obtain

$$\frac{b+\alpha}{2\sqrt{-\Delta}}\ln\frac{-2\beta z+(b-\alpha)-\sqrt{-\xi\Delta}}{-2\beta z+(b-\alpha)+\sqrt{-\Delta}}-\tfrac{1}{2}\ln[a+(b-\alpha)z-\beta z^2]=\ln x+\ln c,$$

which yields

$$x[a+(b-\alpha)z-\beta z^2]^{1/2}\left[\frac{-2\beta z+(b-\alpha)-\sqrt{\xi}}{-2\beta z+(b-\alpha)+\sqrt{\xi}}\right]^{-(b+\alpha)/2\sqrt{\xi}}=c, \quad \text{[A8]}$$

where $\xi=-\Delta$. Using the initial conditions $\tau=0=0$, $v_1'=0$ and $v_2'=0$, we find the integration constant c:

$$c=-\left(\frac{1-\omega}{1+\gamma}\right)^{1/2}\left[\frac{2\gamma_1-(1+\gamma_1)\omega+(1+\gamma_2)-\sqrt{\xi}}{2\gamma_1-(1+\gamma_1)\omega+(1+\gamma_2)+\sqrt{\xi}}\right]^{-(b+\alpha)/2\sqrt{\xi}}. \quad \text{[A9]}$$

APPENDIX B

Let us write the system of linear equations [12] and [13] in the form

$$\frac{dy_1}{d\tau}=a_{11}y_1+a_{12}y_2+f_1 \quad \text{[B1]}$$

and

$$\frac{dy_2}{d\tau}=a_{21}y_1+a_{22}y_2+f_2, \quad \text{[B2]}$$

which is equivalent to the following vector equation:

$$\frac{d\mathbf{Y}}{dt}=A\mathbf{Y}+\mathbf{F}, \quad \text{[B3]}$$

where

$$y_1=\bar{v}_1',\quad y_2=\bar{v}_2',\quad a_{11}=-(1+\gamma_1)\tau_1^{-1},\quad a_{12}=-\gamma_2\tau_1^{-1},\quad f_1=\tau_1^{-1};$$
$$a_{21}=-\gamma_1\tau_2^{-1},\quad a_{22}=-(1+\gamma_2)\tau_2^{-1},\quad f_2=\tau_2^{-1},\quad \tau_1=\frac{d_1^2\rho_2}{18\mu};\quad \tau_2=\frac{d_2^2\rho_2}{18\mu},$$
$$\mathbf{Y}=\begin{pmatrix}y_1\\y_2\end{pmatrix};\quad \mathbf{F}=\begin{pmatrix}f_1\\f_2\end{pmatrix};\quad \mathbf{A}=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}.$$

First of all let us find some particular solution of [B3]. Let us assume that solution $\mathbf{Y}=\mathbf{Y}_p$ does not depend on time τ. Thus, we obtain

$$A\mathbf{Y}_p+\mathbf{F}=\mathbf{0} \quad \text{[B4]}$$

or

$$\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}y_{p1}\\y_{p2}\end{pmatrix}+\begin{pmatrix}f_1\\f_2\end{pmatrix}=0. \quad \text{[B5]}$$

Equation [B5] may be rearranged in the form

$$a_{11}y_{p1}+a_{12}y_{p2}=-f_1,\quad a_{21}y_{p1}+a_{22}y_{p2}=-f_2, \quad \text{[B6]}$$

which yields

$$y_{p1} = \frac{\begin{vmatrix} -f_1 & a_{12} \\ -f_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{-f_1 a_{22} + f_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \quad [B7]$$

and

$$y_{p2} = \frac{\begin{vmatrix} a_{11} & -f_1 \\ a_{21} & -f_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{-a_{11}f_2 + a_{21}f_1}{a_{11}a_{22} - a_{21}a_{12}}. \quad [B8]$$

We see that the determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} -(1+\gamma_1)\tau_1^{-1} & -\gamma_2\tau_1^{-1} \\ -\gamma_1\tau_2^{-1} & -(1+\gamma_2)\tau_2^{-1} \end{vmatrix} = \frac{1+\gamma}{\tau_1\tau_2} \neq 0$$

is not equal to zero. Now let us search for the solution $\mathbf{Y} = \mathbf{Y}_h$ of the homogeneous equation

$$\frac{d\mathbf{Y}}{d\tau} = \mathbf{AY}. \quad [B9]$$

Representing the solution of [B9] in the form

$$\mathbf{Y}_h = \mathbf{C}\,e^{p\tau}, \quad \mathbf{C} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad [B10]$$

we obtain

$$p\mathbf{Y}_h = \mathbf{AY}_h, \quad [B11]$$

which is equivalent to

$$py_{h1} = a_{11}y_{h1} + a_{12}y_{h2}, \quad py_{h1} = a_{21}y_{h1} + a_{22}y_{h2} \quad [B12]$$

and

$$(a_{11} - p)y_{h1} + a_{12}y_{h2} = 0, \quad a_{11}y_{h1} + (a_{22} - p)y_{h2} = 0. \quad [B13]$$

To obtain a nontrivial solution, we need to satisfy the following condition:

$$\begin{vmatrix} a_{11} - p & a_{12} \\ a_{21} & a_{22} - p \end{vmatrix} = 0, \quad a_{11}a_{22} - p(a_{22} + a_{11}) + p^2 - a_{12}a_{21} = 0. \quad [B14]$$

Therefore, from [B14] we find p in the form

$$p_{12} = \frac{a_{11} + a_{22}}{2} \pm \sqrt{\frac{(a_{11} + a_{22})^2}{4} + a_{11}a_{21} - a_{11}a_{22}}. \quad [B15]$$

Then

$$y_{h1}^{(1)} = C_1^{(1)}\exp(p_1\tau), \quad y_2^{(1)} = C_2^{(1)}\exp(p_1\tau), \quad y_{h1}^{(2)} = C_1^{(2)}\exp(p_2\tau); \quad y_{h2}^{(2)} = C_2^{(2)}\exp(p_2\tau). \quad [B16]$$

Substituting [B16] in [B13] we obtain

$$C_1^{(1)} = \frac{a_{22} - p_1}{a_{21}}c_2^{(1)}; \quad C_1^{(2)} = -\frac{a_{22} - p_2}{a_{21}}c_2^{(2)}. \quad [B17]$$

Therefore, the solutions of the homogeneous equation [B9] are as follows:

$$\mathbf{Y}_{h1} = \begin{bmatrix} -\frac{(a_{22} - p_1)}{a_{21}} \cdot C_2^{(2)} \\ C_2^{(1)} \end{bmatrix} \exp(p_1\tau); \quad \mathbf{Y}_{h2} = \begin{bmatrix} -\frac{(a_{22} - p_2)}{a_{21}} \cdot C_2^{(2)} \\ C_2^{(2)} \end{bmatrix} \exp(p_2\tau). \quad [B18]$$

Hence,

$$\mathbf{Y}_{\mathrm{h}} = A\begin{bmatrix} -\dfrac{(a_{22}-p_1)}{a_{21}}\exp(p_1\tau) \\ \exp(p_1\tau) \end{bmatrix} + B\begin{bmatrix} -\dfrac{(a_{22}-p_2)}{a_{21}}\exp(p_2\tau) \\ \exp(p_2\tau) \end{bmatrix}, \qquad [B19]$$

where $A = C_2^{(1)}$ and $B = C_2^{(2)}$. Thus, the solution of the initial equation [B3] may be obtained by using [B7], [B8] and [B19] in the following form:

$$y_1 = -A\frac{(a_{22}-p_1)}{a_{21}}\exp(p_1\tau) - B\frac{(a_{22}-p_2)}{a_{21}}\exp(p_2\tau) + \frac{-f_1a_{21}+f_2a_{12}}{a_{11}a_{22}-a_{21}a_{12}} \qquad [B20]$$

and

$$y_2 = A\exp(p_1\tau) + B\exp(p_2\tau) + \frac{-a_{11}f_2 + a_{21}f_1}{a_{11}a_{22}-a_{21}a_{11}}. \qquad [B21]$$

Using the initial condition $\tau = 0$ and $y_1 = y_2 = 0$, we obtain the equations for the constants A and B:

$$0 = -A\frac{a_{22}-p_1}{a_{21}} - B\frac{a_{22}-p_2}{a_{21}} + \frac{-f_1a_{22}+f_2a_{12}}{a_{11}a_{22}-a_{22}a_{12}} \qquad [B22]$$

and

$$0 = A + B + \frac{-a_{11}f_2 + a_{22}f_1}{a_{11}a_{22}-a_{21}a_{12}}, \qquad [B23]$$

which yield

$$A = \epsilon\left[\frac{a_{22}-p_1}{a_{21}} + 1\right]\frac{a_{21}}{p_2-p_1} - \epsilon$$

and

$$B = -\epsilon\left[\frac{a_{22}-p_1}{a_{21}} + 1\right]\frac{a_{21}}{p_2-p_1},$$

where $\epsilon = (1+\gamma)^{-1}$. Now we are able to write the expressions for $\bar{v}_1'$ and $\bar{v}_2'$:

$$\frac{\bar{v}_1'}{\epsilon} = -\left[\left(\frac{a_{22}-p_1}{a_{21}}+1\right)\frac{a_{21}}{p_2-p_1} - 1\right]\frac{a_{22}-p_1}{a_{21}}\exp(p_1\tau) + \left(\frac{a_{22}-p_1}{a_{21}}+1\right)\frac{a_{21}}{p_2-p_1}\cdot\frac{a_{22}-p_2}{a_{21}}\exp(p_2\tau) + 1 \qquad [B24]$$

and

$$\frac{\bar{v}_2'}{\epsilon} = \left[\left(\frac{a_{22}-p_1}{a_{21}}+1\right)\frac{a_{21}}{p_2-p_1} - 1\right]\exp(p_1\tau) - \left(\frac{a_{22}-p_1}{a_{21}}+1\right)\frac{a_{21}}{p_2-p_1}\exp(p_2\tau) + 1. \qquad [B25]$$

These expressions may be rearranged in the following form:

$$\bar{v}_1' = +(1+\gamma)^{-1}\{-(\varphi_2-\varphi_1)^{-1}[\gamma_1^{-1}(1+\gamma)-\varphi_2][\gamma_1^{-1}(1+\gamma_2)-\varphi_1]\exp(-\gamma_1\varphi_1\tau^*) + (\varphi_2-\varphi_1)^{-1}[\gamma_1^{-1}(1+\gamma)-\varphi_1][\gamma_1^{-1}(1+\gamma_2)-\varphi_2]\exp(-\gamma_1\varphi_2\tau^*) + 1\} \qquad [B26]$$

and

$$\bar{v}_2' = (1+\gamma)^{-1}\{(\varphi_2-\varphi_1)^{-1}[\gamma_1^{-1}(1+\gamma)-\varphi_2]\exp(-\gamma_1\varphi_1\tau^*) - (\varphi_2-\varphi_1)^{-1}[\gamma_1^{-1}(1+\gamma)-\varphi_1]\exp(-\gamma_1\varphi_2\tau^*) + 1\}. \qquad [B27]$$

where

$$\varphi_{1,2} = \left[\frac{a_{11}+a_{22}}{a_{21}} \pm \sqrt{\left(\frac{a_{11}-a_{22}}{a_{21}}\right)^2 + 4\frac{a_{12}}{a_{21}}}\right]\cdot 0.5.$$

APPENDIX C

We estimate the values of the regime parameters corresponding to the assumption that the particles do not leave the fluid element as this fluid element persists as an entity.

The particle moving with relative $\tilde{v}$ during the interaction time t^* passes the distance S inside the fluid element:

$$S = \int_0^{t^*} \tilde{v}\,\mathrm{d}t. \qquad [C1]$$

Therefore, the assumption under consideration corresponds to the following condition:

$$\int_0^{t^*} \tilde{v}\,\mathrm{d}t \leqslant d_f, \qquad [C2]$$

where d_f is a characteristic size of the fluid element. Integrating [9] under the initial condition $t = 0$ and $v'_p = 0$, we arrive at

$$v'_p = (1+\gamma)^{-1}\left\{1 - \exp\left[-\frac{t}{\tau}(1+\gamma)\right]\right\} \qquad [C3]$$

and

$$\bar{\tilde{v}} = \exp\left[-\frac{t}{\tau}(1+\gamma)\right]. \qquad [C4]$$

By using the mass balance equation for a fluid element with particles (for a monodisperse dilute mixture), we obtain

$$d_f = d_p\left(\frac{\bar{\rho}}{\gamma}\right)^{1/3} \qquad [C5]$$

Substitution of [C4] and [C5] in [C2] yields

$$\frac{d_p v'_0}{U}\{1 - \exp[-\tau^*(1+\gamma)]\} \leqslant \frac{18(1+\gamma)}{(\bar{\rho}^2\gamma)^{1/3}}. \qquad [C6]$$

Inequality [C6] is satisfied in a number of real two-phase flows. For example, in a submerged particle-laden gas jet ($\gamma = 0\cdot 1, d_p = 5\cdot 10^{-5}\,\mathrm{m}, \bar{\rho} = 8\cdot 10^2, \nu = 10^{-5}\,\mathrm{m^2/s}, v'_0 = 1\,\mathrm{m/s}, t^* = 10^{-3}\,\mathrm{s}$) we have:

$$\frac{d_p v'_0}{\nu}\{1 - \exp[-\tau^*(1+\gamma)]\} = 0.474 < \frac{18(1+\gamma)}{(\bar{\rho}^2\gamma)^{1/3}} = 0.495.$$